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Interaction of radiation with atoms

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Abstract. A method utilising the coherent state variables for the electromagnetic field and appropriate spin variables for two-level atoms is developed for obtaining the dynamical evolution of the field and the atoms separately. The case of a single radiation mode interacting with two two-level atoms is studied. The time evolution of the photon number, starting from n photons and both atoms at their higher levels, is given. Furthermore, when the initial state consists of n photons and one of the atoms in its excited state with the other in its ground state, the temporal development of the photon number, as well as the motion of each atom, are obtained. In the latter case it becomes apparent that energy exchange takes place among the atomic systems, while the field acts as a transfer agent.

1. Introduction

This paper is concerned with a method for handling problems of quantised radiation in a cavity interacting with matter in the case of two-level atoms, represented by spins $\frac{1}{2}$. The case of a single classical radiation mode interacting with a spin $\frac{1}{2}$ was originally treated by Rabi (1937) exactly within the rotating-wave approximation. Within the same approximation the fully quantum-mechanical treatment was given by Jaynes and Cummings (1963), Cummings (1965) and Scully and Lamb (1967) by use of photon number representation. Further, Pike and Swain (1970, 1971) dealt with a single- and multi-mode radiation in connection with non-equilibrium quantum statistics.

The multi-atom case interacting with a single radiation mode was treated by Tavis and Cummings (1968). These authors obtained the eigenvalues and expressions for the eigenvectors of the irreducible form of the problem. Their expressions required numerical treatment. To the extent of finding the eigenvalues and eigenvectors in the irreducible representation, in a completely analytic fashion involving a few two-level atoms interacting with a single mode of radiation at resonance with the atoms, these were given by Mallory (1969). Mallory used the coherent state representation for the electromagnetic field and the various spin values for the atomic states.

However, if one wishes to obtain the temporal development of the photon number in a cavity, containing the atomic system, the information supplied by the eigenvalues and the eigenvectors of the irreducible representation needs to be processed further. In fact, the full field atomic system propagator is necessary to answer the question of how the photon number evolves in time, starting from a given photon number and a given atomic system state at an initial time.

Walls (1971a, b) dealt with the initial-value problem, obtaining probabilities of interest. The method employed, under certain circumstances, possesses a good deal of

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simplifying features, and Walls applied it to a variety of special cases. The method presented in this paper, although conveyed through a particular case, is general and can supply the mean values of quantities of interest as well as the associated probabilities. It is of particular interest when one is asking questions requiring use of a great deal of the propagator.

In this work we develop a method leading to the complete propagator of the problem, aimed for use with the initial-value problem. We work out the case of a single radiation mode interacting with two two-level atoms in a way prescribed by the well-known model Hamiltonian:

$$H = \hbar\{\omega a^+ a + \omega(S_1^+ S_1 + S_2^+ S_2) + g a^+ (S_1 + S_2) + g a (S_1^+ + S_2^+)\} \tag{1.1}$$

where a^+, a represent the usual photon creation and annihilation operators and S^+, S are related to the Pauli spin matrices via $\sigma_x = S + S^+$, $\sigma^y = \mathrm{i}(S - S^+)$.

We employ an appropriate system of spin-photon states forming the basis of the method which is naturally extended to include the multi-mode case as well as the situation where many atoms are involved. Furthermore the various elements of the propagator are easily accessible to interpretation.

It would seem appropriate at this stage to review the spin states we use in this work. Essentially they form a modification (to account for the spin statistics) of the Fermi states employed earlier by Papadopoulos (1976) in connection with functional integrals for fermions.

For a system of N $\frac{1}{2}$ spins we associate a state labelled by N complex numbers σ_j, as

$$|\sigma_N, \ldots, \sigma_2, \sigma_1\rangle = \pi^{-N/2} \exp\Big(-\tfrac{1}{2}\sum_{j=1}^{N} |\sigma_j|^2\Big)\Big[\prod_{j=1}^{N} (1 + \sigma_j S_j^+)\Big]\,|0\rangle \tag{1.2}$$

where $|0\rangle$ is the spin vacuum state (all spins down). The bra form of (1.2a) is

$$\langle\sigma_1, \sigma_2, \ldots, \sigma_N| = \pi^{-N/2} \exp\Big(-\tfrac{1}{2}\sum_{j=1}^{N} |\sigma_j|^2\Big)\langle 0|\prod_{j=1}^{N} (1 + \bar{\sigma}_j S) \tag{1.2a}$$

where $\bar{\sigma}$ denotes the complex conjugate of σ.

The operators S_j^+, S_j, as is well known, obey fermion properties for the same spin:

$$S_j^+ S_j + S_j^+ S_j = 1, \qquad S_j^{+2} = S_j^2 = 0,$$

while for different spins ($j \neq l$) they obey boson properties.

$$[S_j, S_l^+] = [S_j^+, S_l^+] = [S_j, S_l] = 0.$$

It is on account of the boson-like portion of the spin operator properties that we are able to use the product $\Pi(1 + \sigma_j S_j^+)$ irrespective of the order of the various factors involved, a facility lacking in the fermion case.

It should be noted here that the spin states (1.2a, b) are in no way coherent states for clearly they are not eigenstates of the lowering operators S_j. Nevertheless they form a complete set:

$$\int |\sigma_N, \ldots, \sigma_2, \sigma_1\rangle\langle\sigma_1, \sigma_2, \ldots, \sigma_N| \prod_{j=1}^{N} \mathrm{d}^2\sigma_j = 1 \qquad (\mathrm{d}^2\sigma = \mathrm{d}\,\mathrm{Re}\,\sigma\,\mathrm{d}\,\mathrm{Im}\,\sigma) \tag{1.3}$$

in the sense that the operator 1 on the RHS of (1.3) stands for the sum of all combinations of discrete terms of the form $|0\rangle\langle 0|$, $|\downarrow_N \ldots \downarrow_2 \uparrow_1\rangle\langle\uparrow_1 \downarrow_2 \ldots \downarrow_N|$ and all possibilities with one spin up, two spins up, ..., $|\uparrow_N \ldots \uparrow_2 \uparrow_1\rangle\langle\uparrow_1 \uparrow_2 \ldots \uparrow_N|$.

One can say that the continuous nature of the labels σ in the states (1.2) makes their totality overcomplete in that they represent states over and above the discrete ones. However, what is important is that they obey the correct completeness relation (1.3), and this is achieved through the exponential factors $\exp(-\frac{1}{2}\Sigma|\sigma_j|^2)$, which enable exploitation of the integrals:

$$\int \pi^{-1} \exp(-|\sigma|^2) \begin{pmatrix} 1 \\ |\sigma|^2 \\ \bar{\sigma}, \sigma \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0, 0 \end{pmatrix} \tag{1.4}$$

which is all one needs to get rid of the non-discrete quantities. The same phenomenon appears in the coherent states, and the main advantage deriving from the use of continuously labelled states is the ability to cast the Schrödinger equation in an integral form. By the way, the use of complex variables supplies the highest degree of symmetry in our expressions.

It should be noted here that, as with the coherent states, the above spin states are not orthogonal. Furthermore, they are normalised by the integral

$$\int \langle \sigma_1, \sigma_2, \ldots, \sigma_N | \sigma_N, \ldots, \sigma_2, \sigma_1 \rangle \prod_{j=1}^{N} \mathrm{d}^2\sigma_j = 2^N \tag{1.5}$$

which gives the total number of states of N spins. For more details concerning the above spin states see Papadopoulos (1978).

We turn now to the question of spin photon states appropriate for problems involving Hamiltonians composed of spin operators, and photon creation and annihilation operators. We take for the system of these states all possible products of the coherent states of the problem with the corresponding spin states. In particular, the states associated with our Hamiltonian (1.1) will be of the form $|\sigma_2, \sigma_1\rangle|\alpha\rangle$, denoted by $|\sigma_2, \sigma_1; \alpha\rangle$, and where $|\alpha\rangle$ stands for a single-species photon coherent state.

Now, the propagator associated with our Hamiltonian is obtained as a matrix element of the corresponding evolution operator as

$$\langle \alpha; \sigma_1, \sigma_2 | U(t) | \sigma_2', \sigma_1'; \alpha' \rangle = \langle \alpha; \sigma_1, \sigma_2 | \exp(-\mathrm{i}Ht/\hbar) | \sigma_2', \sigma_1'; \alpha' \rangle. \tag{1.6}$$

Schrödinger's equation for the propagator takes the form:

$$\begin{aligned} \mathrm{i}\hbar \frac{\partial}{\partial t} \langle \alpha; \sigma_1, \sigma_2 | U(t) | \sigma_2', \sigma_1'; \alpha' \rangle \\ = \int \langle \alpha; \sigma_1, \sigma_2 | H | \sigma_2'', \sigma_1''; \alpha'' \rangle \langle \alpha''; \sigma_1'', \sigma_2'' | U(t) | \sigma_2', \sigma_1'; \alpha' \rangle \, \mathrm{d}^2\sigma_1'' \, \mathrm{d}^2\sigma_2'' \, \mathrm{d}^2\alpha''. \end{aligned} \tag{1.7}$$

In order to avoid interrupting the text later we quote here the following identities concerning the coherent state variables:

$$\int \exp(\bar{\alpha}\alpha' - |\alpha'|^2) A(\bar{\alpha}') \begin{pmatrix} 1 \\ \alpha' \end{pmatrix} \frac{\mathrm{d}^2\alpha'}{\pi} = \begin{pmatrix} A(\bar{\alpha}) \\ \dfrac{\partial}{\partial \bar{\alpha}} A(\bar{\alpha}) \end{pmatrix}. \tag{1.8}$$

These relations can be viewed from the representation point of the operators a^+, a as follows: a^+ replaced by $\bar{\alpha}$ and a by $\partial/\partial\bar{\alpha}$. Clearly, these expressions satisfy the correct commutation relations for Bose operators, i.e. $[\partial/\partial\bar{\alpha}, \bar{\alpha}] = 1$. This representation was also derived by Mallory (1969) starting from an alternative realisation of the field operators.

In § 2 we apply our method for obtaining the propagator for a single mode of radiation interacting with two two-level atoms. The result is used for solving the initial-value problem for finding the photon number evolution in a cavity in which we start with n photons while the two atoms are excited to their upper levels. Furthermore, the dynamical evolution of the radiation, and each of the atoms separately, is obtained when we start with n photons and one of the atoms is in its upper level, while the other is in its ground state. In all cases the radiation is in tune with the atoms.

2. One mode of radiation interacting with two atoms

Having laid down preliminary work in the previous section we now proceed to obtain the propagator associated with the Hamiltonian (1.1), thus acquiring what one needs to solve the initial-value problem for both the radiation and the atomic dynamics separately.

Let us now write down the matrix element of the Hamiltonian (1.1) in the $\sigma_1, \sigma_2; \alpha$ representation. We have:

$$\begin{aligned}\langle \alpha; \sigma_1, \sigma_2 | H | \sigma_2', \sigma_1'; \alpha' \rangle &= \hbar[\omega\bar{\alpha}\alpha' + (\omega + \omega\bar{\alpha}\alpha')(\bar{\sigma}_1\sigma_1' + \bar{\sigma}_2\sigma_2') + (2\omega + \omega\bar{\alpha}\alpha')\bar{\sigma}_1\bar{\sigma}_2\sigma_1'\sigma_2' \\ &\quad + g\bar{\alpha}(\sigma_1' + \sigma_2' + \bar{\sigma}_1\sigma_1'\sigma_2' + \bar{\sigma}_2\sigma_1'\sigma_2') + g\alpha'(\bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_1\bar{\sigma}_2\sigma_1' + \bar{\sigma}_1\bar{\sigma}_2\sigma_2')] \\ &\quad \times \pi^{-3} \exp\left[\bar{\alpha}\alpha' - \tfrac{1}{2}(|\alpha|^2 + |\alpha'|^2) - \tfrac{1}{2}\sum_{j=1}^{2}(|\sigma_j|^2 + |\sigma_j'|^2)\right]. \end{aligned} \tag{2.1}$$

The propagator in this representation will be

$$\begin{aligned}\langle \alpha; \sigma_1, \sigma_2 | U(t) | \sigma_2', \sigma_1'; \alpha' \rangle &= (U_{00} + U_{01}\sigma_1' + U_{02}\sigma_2' + U_{03}\sigma_1'\sigma_2' + U_{10}\bar{\sigma}_1 + U_{11}\bar{\sigma}_1\sigma_1' + U_{12}\bar{\sigma}_1\sigma_2' \\ &\quad + U_{13}\bar{\sigma}_1\sigma_1'\sigma_2' + U_{20}\bar{\sigma}_2 + U_{21}\bar{\sigma}_2\sigma_1' + U_{22}\bar{\sigma}_2\sigma_2' + U_{23}\bar{\sigma}_2\sigma_1'\sigma_3' + U_{30}\bar{\sigma}_1\bar{\sigma}_2 \\ &\quad + U_{31}\bar{\sigma}_1\bar{\sigma}_2\sigma_1' + U_{32}\bar{\sigma}_1\bar{\sigma}_2\sigma_2' + U_{33}\bar{\sigma}_1\bar{\sigma}_2\sigma_1'\sigma_2') \\ &\quad \times \pi^{-3} \exp\left[\bar{\alpha}\alpha' - \tfrac{1}{2}(|\alpha|^2 + |\alpha'|^2) - \tfrac{1}{2}\sum_{j=1}^{2}(|\sigma_j|^2 + |\sigma_j'|^2)\right] \end{aligned} \tag{2.2}$$

where in (2.2) $U_{ij} = U_{ij}(\bar{\alpha}, \alpha', t)$ and have to be suitably evaluated.

Inserting (2.2) for the propagator and (2.1) for the Hamiltonian into the Schrödinger integral equation (1.7) and after making use of the identity (1.4) and equating the coefficients of the same products of $\bar{\sigma}$ and σ' we convert our integral equation into a set of simultaneous partial differential equations which, in matrix notation, take the form:

$$i\hbar\frac{\partial}{\partial t}\mathscr{V} = \mathscr{H}\mathscr{V} \tag{2.3}$$

where $\mathscr{H}$ and $\mathscr{V}$ are given by matrices:

$$\frac{1}{\hbar}\mathscr{H}=\begin{pmatrix} \omega\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}} & g\bar{\alpha} & g\bar{\alpha} & 0 \\ g\frac{\partial}{\partial\bar{\alpha}} & \omega\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}+\omega & 0 & g\bar{\alpha} \\ g\frac{\partial}{\partial\bar{\alpha}} & 0 & \omega\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}+\omega & g\bar{\alpha} \\ 0 & g\frac{\partial}{\partial\bar{\alpha}} & g\frac{\partial}{\partial\bar{\alpha}} & \omega\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}+2\omega \end{pmatrix}. \tag{2.3a}$$

This is a (4×4) matrix, and so will be the propagator put in matrix formation:

$$\mathscr{V}(\bar{\alpha}, a', t)=\begin{pmatrix} U_{00} & U_{10} & U_{02} & U_{03} \\ U_{10} & U_{11} & U_{12} & U_{13} \\ U_{20} & U_{21} & U_{22} & U_{23} \\ U_{30} & U_{31} & U_{32} & U_{33} \end{pmatrix}. \tag{2.3b}$$

The simultaneous equations derived from (2.3) are the equations of motion for the various propagator components, entirely dependent on the coherent variables and the time. They must be solved under the initial conditions:

$$U_{ij}(\bar{\alpha}, \alpha', 0)=\delta_{ij}\exp(\bar{\alpha}\alpha')=\delta_{ij}\sum_{n=0}^{\infty}\frac{\bar{\alpha}^n\alpha'^n}{n!}. \tag{2.3c}$$

These conditions derive immediately from the fact that the evolution operator at $t=0$ is $U(0)=1$, and therefore the propagator equals the product:

$$\begin{aligned}&\langle\sigma_1\sigma_2|\sigma_2'\sigma_1'\rangle\langle\alpha|\alpha'\rangle \\ &\quad=\langle 1+\bar{\sigma}_1\sigma_1'+\bar{\sigma}_2\sigma_2'+\bar{\sigma}_1\bar{\sigma}_2\sigma_1'\sigma_2'\rangle \\ &\qquad\times\pi^{-3}\exp\left[\bar{\alpha}\alpha'-\tfrac{1}{2}(|\alpha|^2+|\alpha'|^2)-\tfrac{1}{2}\sum_{j=1}^{2}(|\sigma_j|^2+|\sigma_j'|^2)\right].\end{aligned} \tag{2.4}$$

We now write down the simultaneous equations for the first column:

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}U_{00}&=\omega\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}U_{00}+g\bar{\alpha}(U_{10}+U_{20}) \\ \mathrm{i}\frac{\partial}{\partial t}U_{10}&=g\frac{\partial}{\partial\bar{\alpha}}U_{00}+\omega\left(\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}+1\right)U_{10}+g\bar{\alpha}U_{30} \\ \mathrm{i}\frac{\partial}{\partial t}U_{20}&=g\frac{\partial}{\partial\bar{\alpha}}U_{00}+\omega\left(\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}+1\right)U_{20}+g\bar{\alpha}U_{30} \\ \mathrm{i}\frac{\partial}{\partial t}U_{30}&=g\frac{\partial}{\partial\bar{\alpha}}(U_{10}+U_{20})+\omega\left(\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}+2\right)U_{30}.\end{aligned} \tag{2.5}$$

Having in mind the initial conditions (2.3*c*) and taking account of (2.5) we look for solutions in the form:

$$U_{00}=\sum_{n=0}^{\infty}u_{00}^{(n)}(t)\frac{\bar{\alpha}^{n}\alpha'^{n}}{n!} \qquad U_{10}=\sum_{n=1}^{\infty}u_{10}^{(n)}(t)\frac{\bar{\alpha}^{n-1}\alpha'^{n}}{n!}$$

$$U_{20}=\sum_{n=1}^{\infty}u_{20}^{(n)}(t)\frac{\bar{\alpha}^{n-1}\alpha'^{n}}{n!} \qquad U_{30}=\sum_{n=2}^{\infty}u_{20}^{(n)}(t)\frac{\bar{\alpha}^{n-2}\alpha'^{n}}{n!}. \tag{2.5a}$$

Inserting the forms (2.5*a*) in (2.5) we obtain from the resulting equations (in the form of power series in $\bar{\alpha}$ and α') the equations governing the motion of the various coefficients $u^{(n)}$. If, in the process of the above derivation, we add and subtract the second and third equations in (2.5) we have:

$$\begin{aligned}
&\mathrm{i}\dot{u}_{00}^{(n)}=\omega n u_{00}^{(n)}+g(u_{10}^{(n)}+u_{20}^{(n)})\\
&\mathrm{i}(\dot{u}_{10}^{(n)}+\dot{u}_{20}^{(n)})=2gnu_{00}^{(n)}+\omega n(u_{10}^{(n)}+u_{20}^{(n)})+2gu_{30}^{(n)}\\
&\mathrm{i}\dot{u}_{30}^{(n)}=g(n-1)(u_{10}^{(n)}+u_{20}^{(n)})+\omega n u_{30}^{(n)}
\end{aligned} \tag{2.5b}$$

$$\mathrm{i}(\dot{u}_{10}^{(n)}-\dot{u}_{20}^{(n)})=\omega n(u_{10}^{(n)}-u_{20}^{(n)}). \tag{2.5c}$$

In order to obtain the required solution for the first column we apply routine normal-mode analysis to (2.5*a*) and make use of (2.5*c*), under the initial conditions

$$u_{00}^{(n)}(0)=1 \qquad u_{10}^{(n)}(0)=u_{20}^{(n)}(0)=u_{30}^{(n)}(0)=0 \tag{2.5d}$$

derived from (2.3*c*).

The eigenvalues associated with the system (2.5*b*) are:

$$\lambda_0^{(n)}=\omega n \qquad \lambda_1^{(n)}=\omega n+\Omega_n \qquad \lambda_2^{(n)}=\omega n-\Omega_n \tag{2.6}$$

where $\Omega_n=g[2(2n-1)]^{1/2}$.

The eigenvalues associated with the other columns are $\lambda_k^{(n+1)}$ $(k=0,1,2)$ for the second and third columns, and $\lambda_k^{(n+2)}$ $(k=0,1,2)$ for the fourth column. In addition the corresponding equations of motion for their various $u^{(n)}$ components are those for the first column (2.5*b*, *c*), but with the n in the coefficients of the unknowns replaced by $(n+1)$ for the second and third column equations, and by $(n+2)$ for the fourth column equations. However, because of the differing initial conditions for each column, the solution cannot be obtained in a fully recursive fashion. Nevertheless, there is a facility in the case of the second and third columns, corresponding to the same physical situation. This is the instance of one spin up and it does not matter, of course, which of the two is up.

The required $u^{(n)}(t)$ for the first column are:

$$\begin{aligned}
u_{00}^{(n)}(t)&=\frac{1}{2n-1}\exp(-\mathrm{i}\omega nt)[n\cos(\Omega_n t)+(n-1)]\\
u_{10}^{(n)}(t)=u_{20}^{(n)}(t)&=\frac{-\mathrm{i}n\Omega_n}{2g(2n-1)}\exp(-\mathrm{i}\omega nt)\sin(\Omega_n t)\\
u_{30}^{(n)}(t)&=\frac{n(n-1)}{2n-1}\exp(-\mathrm{i}\omega nt)[\cos(\Omega_n t)-1].
\end{aligned} \tag{2.6a}$$

Working in a similar fashion we find the expressions for the rest of the propagator elements. The components of the second and third columns can be obtained from knowledge of one of them, by interchanging the two middle ones. To avoid duplication we write the expressions for the second and third columns together. We have:

$$U_{01} = \sum u_{01}^{(n)} \frac{\bar{\alpha}^{n+1}\alpha'^{n}}{n!} = U_{02} = \sum u_{02}^{(n)} \frac{\bar{\alpha}^{n+1}\alpha'^{n}}{n!}$$

$$U_{11} = \sum u_{11}^{(n)} \frac{\bar{\alpha}^{n}\alpha'^{n}}{n!} = U_{22} = \sum u_{22}^{(n)} \frac{\bar{\alpha}^{n}\alpha'^{n}}{n!}$$

$$U_{21} = \sum u_{21}^{(n)} \frac{\bar{\alpha}^{n}\alpha'^{n}}{n!} = U_{12} = \sum u_{12}^{(n)} \frac{\bar{\alpha}^{n}\alpha'^{n}}{n!}$$

$$U_{31} = \sum u_{31}^{(n)} \frac{\bar{\alpha}^{n-1}\alpha'^{n}}{n!} = U_{32} = \sum u_{32}^{(n)} \frac{\bar{\alpha}^{n-1}\alpha'^{n}}{n!} \tag{2.7}$$

where

$$u_{01}^{(n)}(t) = u_{02}^{(n)}(t) = \frac{-ig}{\Omega_{n+1}} \exp[-i\omega(n+1)t] \sin(\Omega_{n+1}t)$$

$$u_{11}^{(n)}(t) = u_{22}^{(n)}(t) = \tfrac{1}{2} \exp[-i\omega(n+1)t][\cos(\Omega_{n+1}t) + 1]$$

$$u_{21}^{(n)}(t) = u_{12}^{(n)}(t) = \tfrac{1}{2} \exp[-i\omega(n+1)t][\cos(\Omega_{n+1}t) - 1]$$

$$u_{31}^{(n)}(t) = u_{13}^{(n)}(t) = \frac{-ing}{\Omega_{n+1}} \exp[-i\omega(n+1)t] \sin(\Omega_{n+1}t). \tag{2.7a}$$

Finally the components of the fourth column are given by:

$$U_{03} = \sum u_{03}^{(n)} \frac{\bar{\alpha}^{n+2}\alpha'^{n}}{n!}$$

$$U_{13} = \sum u_{13}^{(n)} \frac{\bar{\alpha}^{n+1}\alpha'^{n}}{n!} = U_{23} = \sum u_{23}^{(n)} \frac{\bar{\alpha}^{n+1}\alpha'^{n}}{n!}$$

$$U_{33} = \sum u_{33}^{(n)} \frac{\bar{\alpha}^{n}\alpha'^{n}}{n!} \tag{2.8}$$

where

$$u_{03}^{(n)}(t) = \frac{1}{2n+3} \exp[-i\omega(n+2)t][\cos(\Omega_{n+2}t) - 1]$$

$$u_{13}^{(n)}(t) = u_{23}^{(n)}(t) = \frac{-i\Omega_{n+2}}{2g(2n+3)} \exp[-i\omega(n+2)t] \sin(\Omega_{n+2}t)$$

$$u_{03}^{(n)}(t) = \frac{1}{2n+3} \exp[-i\omega(n+2)t][(n+1)\cos(\Omega_{n+2}t) + (n+2)]. \tag{2.8a}$$

Expressions (2.5*a*), (2.6*a*), (2.7), (2.7*a*), (2.8) and (2.8*a*) complete the evaluation of the propagator.

We now have at our disposal all of what we need to answer questions such as, what is the development in time of the photon number or the state of each atom separately, given the photon number and the state of each atom at an initial time.

As a first application let us make use of our propagator for obtaining the evolution in time of the photon number of a single mode of radiation interacting with two spins, each flipping at an energy equal to that of the photon species involved. We begin with an initial state of n photons present in the cavity and with both spins in their higher levels. In terms of the radiation and spin flipping operators the initial state is $(n!)^{-1/2}a^{+n}S_1^+S_2^+|0\rangle$. The temporal development of the photon number is then given by:

$$\langle a^+a\rangle_t = \langle 0|S_2S_1\frac{a^n}{\sqrt{n!}}\exp\left(\frac{\mathrm{i}}{\hbar}Ht\right)a^+a\exp\left(-\frac{\mathrm{i}}{\hbar}Ht\right)\frac{a^{+n}}{\sqrt{n!}}S_1^+S_2^+|0\rangle. \tag{2.9}$$

Using the representation $\sigma_1, \sigma_2; \alpha$ in which our propagator has been expressed we find, after performing the various α and σ integrations involved, the result:

$$\langle a^+a\rangle_t = (n+2)u_{30}^{(n+2)}(-t)u_{03}^{(n)}(t) + 2(n+1)u_{31}^{(n+1)}(-t)u_{13}^{(n)}(t) + nu_{33}^{(n)}(-t)u_{33}^{(n)}(t). \tag{2.9a}$$

In deriving (2.9a) we have taken account of the equalities $u_{31}^{(n)} = u_{32}^{(n)}$ and $u_{13}^{(n)} = u_{23}^{(n)}$. Further utilising the expressions for the various coefficients $u^{(n)}$ from (2.6a), (2.7) and (2.7a) we find:

$$\langle a^+a\rangle_t = \frac{1}{(2n+3)^2}[(2n+3)(2n^2+5n+3) - 1 - (n+1)\sin^2(\Omega_{n+2}t) - 4(n+1)(n+2)\cos(\Omega_{n+2}t)]. \tag{2.9b}$$

We believe (2.9b) to be a new analytic result concerning the initial-value problem in the case of two atoms. The result involving a single atom can be found, for example, in Cummings (1965).

We recall that the quantity $\Omega_{n+2} = g[2(2n+3)]^{1/2}$ is approximately twice the flipping frequency, $g(n+1)^{1/2}$ for the case when one spin is involved. In the case of one atom the photon number rises from n to $n+1$ and falls back to n periodically with a period $\pi/g(n+1)^{1/2}$.

We would like to look into the analogous situation when two atoms are involved. From (2.9b) it follows that the photon number is an oscillatory function of time (see figure 1) with its maxima and minima (obtained by differentiation with respect to t) occurring at the moments $t_k = k\pi/2g(n+3/2)^{1/2}$ $(k = 0, 1, 2, \ldots)$. Each minimum equals n, when k is even, while each maximum equals $[n+2-2/(2n+3)^2]$, which is not quite $n+2$, when k is odd. This analysis shows that the photon number oscillates, now, with a period $\pi/g(n+3/2)^{1/2}$, which is approximately the same as for the single-atom case, the difference being in the light intensity. Owing to the fact that the maximum of the photon number cannot become exactly $n+2$, although initially the two atoms were charged exactly with two photons, it would appear that as long as the photon numbers involved are small it is not easy to have them both simultaneously released. This is in contrast with the situation involving one atom.

The spontaneous emission is obtained from (2.9a) by taking $n = 0$. We have:

$$\langle a^+a\rangle_{t,\text{spon.}} = \tfrac{8}{9}[1 - \cos(g\sqrt{6}t) - \tfrac{1}{8}\sin^2(g\sqrt{6}t)]. \tag{2.9c}$$

As a further application let us start with n photons and one of the atoms in its upper level (say atom 1) while the other is in its ground state. In symbols our initial state is $(n!)^{-1/2}a^{+n}S_1^+|0\rangle$. To obtain the time development of the photon number under the

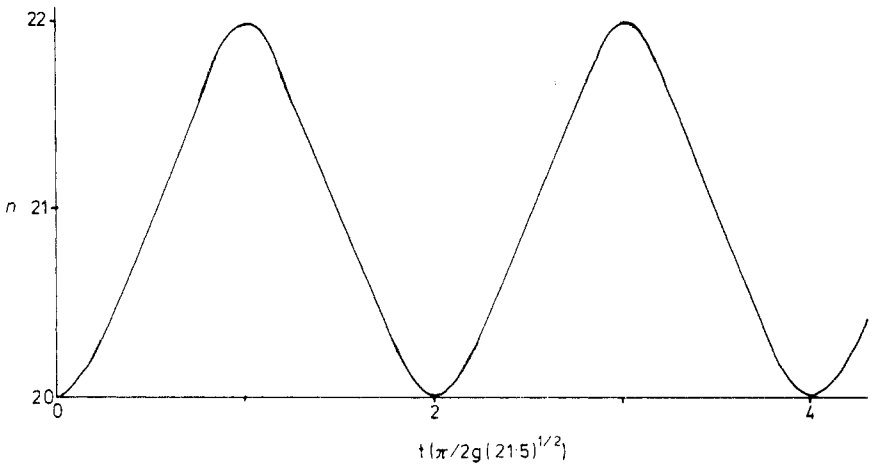

Figure 1. The time evolution of the photon number, when at $t = 0$ there were 20 photons and both atoms are in their excited states. Both atoms expel their energy simultaneously and re-absorb it periodically in a time equal to $\pi/2g(21{\cdot}5)^{1/2}$, half the oscillation period of the photon number.

above initial conditions, we employ (2.9), but now using the new initial state, and we have:

$$\langle a^+a\rangle_t = (n+1)u_{10}^{(n+1)}(-t)u_{01}^{(n)}(t) + n(u_{11}^{(n)}(-t)u_{11}^{(n)}(t) + u_{12}^{(n)}(-t)u_{21}^{(n)}(t)) + (n-1)u_{13}^{(n-1)}(-t)u_{31}^{(n)}(t). \tag{2.10}$$

Again, using in (2.10) expressions from (2.6*a*), (2.7*a*) and (2.8*a*) for the propagator, we find that the photon number in this case will be:

$$\langle a^+a\rangle_t = n + \frac{1}{2(2n+1)}\sin^2(\Omega_{n+1}t). \tag{2.10a}$$

Evidently there is almost no change in the photon number, which at first sight looks somewhat strange, since atom 1 is bound to release a photon at a later moment. The other alternative is that the photon leaving atom 1, instead of entering the radiation pool, will be transferred by the field to atom 2. Then, atom 2 will be in the initial state of atom 1 and the radiation will again act as a transfer agent of energy from atom 2 to 1, and this process of energy exchange between the atoms will continue periodically. We quantify the above picture by evaluating the mean values $\langle S_1^+S_1\rangle_t$ and $\langle S_s^+S_2\rangle_t$, describing the state of occupation of the upper level of the atoms 1 and 2, respectively. To do this we replace successively the operator a^+a in (2.9) by the operators $S_1^+S_1$ and $S_2^+S_2$ and at the same time replace the bra and ket form of the initial state by the corresponding forms of the state $(n!)^{-1/2}a^{+n}S_1^+|0\rangle$. We have:

$$\langle S_1^+S_1\rangle_t = u_{11}^{(n)}(-t)u_{11}^{(n)}(t) = \tfrac{1}{2} + \tfrac{1}{2}\cos(\Omega_{n+1}t) - \tfrac{1}{4}\sin^2(\Omega_{n+1}t) \tag{2.10b}$$

$$\langle S_2^+S_2\rangle_t = u_{12}^{(n)}(-t)u_{21}^{(n)}(t) = \tfrac{1}{2} - \tfrac{1}{2}\cos(\Omega_{n+1}t) - \tfrac{1}{4}\sin^2(\Omega_{n+1}t). \tag{2.10c}$$

The plots of (2.10*b*, *c*) (see figure 2) show clearly the transfer of energy from atom 1 to atom 2. The bottom portions of the curves are flatter than the top ones, showing that the atoms prefer their ground levels. The period of spin flipping is $\pi/g(n+\frac{1}{2})^{1/2}$, approximately the same as with a single spin interacting with the radiation field.

The picture drawn from the above analysis is that when atoms are near each other (their coupling with the radiation is about the same) they exchange energy between themselves, rather than the radiation field, unless the majority are in their excited states.

The present method supplies the field matter propagator in the case of two two-level atoms and a single radiation mode. The extension to a higher number of atoms interacting with multi-mode radiation can be attained in a natural fashion, however, not without an increase in the complexity of the evaluations. The main advantage of the approach is that it provides a systematic way for handling the dynamical evolutions of the radiation and the matter separately.

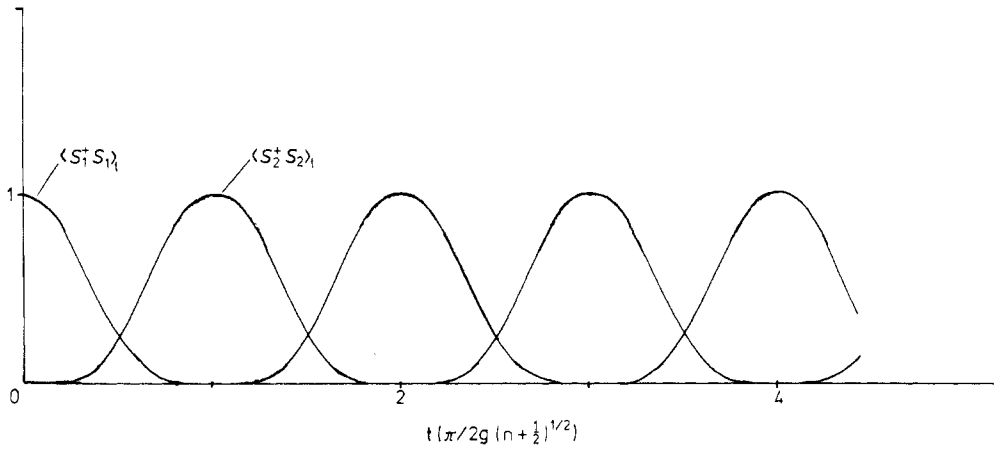

Figure 2. The time development of the state of upper-level occupation for atoms 1 and 2, when at time $t = 0$ there were n photons and atom 1 was in its upper level while atom 2 was in its lower level. As time proceeds the two atoms exchange energy periodically in a time equal to $\pi/2g(n+\frac{1}{2})^{1/2}$, which is half the period of atomic oscillation.

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